How Does Information Quality Affect Stock Returns?

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ABSTRACT

Using a simple dynamic asset pricing model, this paper investigates the relationship between the precision of public information about economic growth and stock market returns. After fully characterizing expected returns and conditional volatility, I show that (i) higher precision of signals tends to increase the risk premium, (ii) when signals are imprecise the equity premium is *bounded above* independently of investors' risk aversion, (iii) return volatility is U-shaped with respect to investors' risk aversion, and (iv) the relationship between conditional expected returns and conditional variance is ambiguous.

IN MODERN FINANCIAL MARKETS, investors are flooded with a variety of information: corporations' earnings reports, revisions of macroeconomic indexes, policymakers' statements, and political news. These pieces of information are processed by investors to update their projections of the economy's future growth rate, inflation rate, and interest rate. In turn, these changes in investors' expectations affect stock market prices. However, even though it is clear that asset prices react to new information, several questions arise regarding the relationship between the *quality* of information that investors receive and asset returns. For example, what kind of effect does a noisy signal on the "health" of the economy have on stock market prices? If information is noisy, is there a risk premium? Or is the risk premium completely independent of the quality of information investors receive? Also, how does the precision of the signals affect stock market volatility? If signals are more precise, does stock market volatility decrease or increase? Finally, can we infer how good investors' information is from the behavior of stock market returns?

In this paper I study a dynamic asset pricing model where I try to answer the above questions. Specifically, I assume that stock dividends are generated by a diffusion process whose drift rate is unknown to investors and may

* University of Chicago. I thank Nick Barberis, Gadi Barlevy, John Cochrane, George Constantinides, Domenico Cuoco, Per Stromberg, Luis Viceira, Francis Yared, and the seminar participants at Harvard University, the University of Chicago, the Università Bocconi (Milano), the 1998 WFA meetings, and the 1998 European Summer Symposium in Financial Markets (Gerzensee, Switzerland) for their comments. I especially thank René Stulz (the editor) and an anonymous referee for their thoughtful suggestions and comments. I am indebted to John Y. Campbell for his guidance on my Ph.D. dissertation, which this paper draws on. All errors are my own.

change at random times. Investors learn about the "true" drift rate through the observation of realized dividends and another noisy signal, which proxies for the many sources of information I refer to above. The main objective of the paper is to characterize equilibrium asset returns when different assumptions on the precision of information—that is, the noise of the signal—are made.

The first surprising result is that more precise signals tend to increase rather than decrease the equity risk premium—that is, there is no risk premium for noisy signals. The converse is also surprising: When signals are noisy, there is an upper bound to the equity premium and this upper bound is independent of investors' degree of risk aversion. Hence, the Mehra and Prescott (1985) equity premium puzzle becomes even more puzzling under the assumption of noisy information because the actual equity premium cannot be matched by assuming a high degree of risk aversion.

To understand the intuition behind these results, consider the second result first. As an extreme and simple case, suppose that dividend realizations are the only signal investors receive about a constant dividend growth rate. Typically then, negative dividend innovations imply a downward revision of expected future dividends and hence of future consumption, because dividends and consumption are highly correlated (in fact, equilibrium market clearing conditions require them to be equal). Hence, risk-averse investors increase their hedging demand for the asset to avoid very low levels of consumption in the future. This latter effect tends to increase the stock price, thereby counterbalancing its tendency to fall due to the initial negative shock to dividends. When investors are sufficiently risk averse, the positive effect on the stock price due to investors' hedging demand for stocks tends to dominate. As a consequence, a drop in current consumption due to a negative innovation in dividends is associated with a small decrease or even an increase in the stock price. That is, in equilibrium the covariance between consumption and returns is small or even negative for high levels of risk aversion. This implies a small or negative risk premium. Indeed, when the coefficient of risk aversion is sufficiently high a further increase in risk aversion decreases the risk premium because of the indirect effect on the covariance of returns and consumption. This implies an upper bound to the equity risk premium.

Turning now to the opposite result—that more precise signals increase the risk premium—consider once again an extreme example. Suppose that investors know exactly the constant drift rate of the economy: In this case innovations in dividends do not change investors' expectations of future dividends. Since a higher dividend implies a higher price for given expectations of future consumption, returns and consumption have positive covariance. This implies a positive equity risk premium. Moreover, just as in the equity premium puzzle literature, a higher coefficient of risk aversion increases the equity risk premium.

The above argument also entails that the precision of signals affects the equilibrium conditional return volatility. Indeed, I show that when signals are imprecise, volatility is first decreasing and then increasing in investors' degree of risk aversion. However, the effect becomes less and less relevant as we increase the precision of signals. The intuition stems again from the hedging demand for the asset: When signals are imprecise, dividend realizations have an impact on investors' hedging demand which tends to decrease the volatility of returns compared to the "dividend" volatility. However, for a sufficiently high risk-aversion coefficient, the indirect effect on the hedging demand dominates increasing return volatility again—hence the U-shaped function of volatility with respect to the coefficient of risk aversion.

An implication of the above discussion is that the relationship between the conditional risk premium and the conditional variance of returns depends on the precision of signals, and this relationship is generally ambiguous: When signals are precise, expected excess returns are positively related to their conditional variance, but the opposite may be true when they are imprecise, depending on the level of investors' uncertainty about the true drift of the economy. This finding helps in explaining the lack of empirical support of a positive relationship between expected excess returns and their conditional variance (see, e.g., Campbell et al. (1999) and Scruggs (1998)).

These results also point at an important difference between current dividend realizations and external signals as predictors of future economic performance. Dividend realizations both change investors' current consumption sets and modify their expectations: Since asset returns depend on changes in expectations through changes in investors' hedging demand for the asset, this dual role of dividends introduces a special covariance between contemporaneous consumption and stock returns, positive or negative depending on investors' preferences. In contrast, external signals only affect expectations and cannot change investors' current consumption sets. As a consequence, when we increase the precision of external signals we are also decreasing the sensitivity of investors' hedging demand to dividend realizations. When signals are perfect, there are no variations in hedging demand due to dividend realizations.

As to the methodology of the paper, I find it useful to discretize the parameter space—that is, the set of possible drifts for the dividend process—to obtain the dynamics of investors' beliefs in closed form. This approach enables me to show that a stock's expected return and volatility depend on a single quantity that summarizes both investors' degree of uncertainty on the true drift of the dividend process and the "relevance" of this uncertainty to asset pricing. For example, this quantity is zero either when investors have perfect information or when the price of the asset is independent of the drift rate of the economy, as in the case where investors have logarithmic utility. In the latter instance of course uncertainty does not matter for asset pricing. On the other hand, this quantity is increasing (in absolute value) both with the "dispersion" of investors' beliefs around the expected growth rate of dividends and the relative difference in asset prices conditional on the various states.

This paper is most closely related to the literature on learning in financial markets. Notable works in this area are Williams (1977), Dothan and Feldman (1986), Gennotte (1986), Detemple (1986, 1991), Feldman (1989), Barsky and DeLong (1993), Timmerman (1993), Wang (1993), Detemple and Murthy (1994), Brennan and Xia (1997), David (1997), Veronesi (1999), and Zapatero (1998). These papers give various characterizations of portfolio allocation rules, term structure models, and stock returns when investors learn about some unknown parameters of the economy. However, none of these papers investigates the issues that I specifically address here—that is, the effect of the precision of external signals on the equilibrium stock return process.

The paper develops as follows: The next section introduces the setup of the simple economy, Section II describes the dynamics of investors' beliefs, and Section III characterizes the stock returns and investors uncertainty. Section IV concludes. All proofs are in the Appendix.

I. The Economy

Consider a standard pure-exchange economy (Lucas (1978)) populated by a continuum of identical investors with isoelastic utility functions,

$$u(c,t) = e^{-\phi t}\,\frac{c^{1-\gamma}}{1-\gamma},$$

where γ is the coefficient of relative risk aversion and ϕ the discount rate. I assume that investors' opportunity set comprises a risky security, whose stochastic dividend[1] is denoted by D, and a bond, whose risk-free rate of return is r. Dividends grow according to the following process:

$$dD = \theta D dt + \sigma_D D dB_D,$$

where B_D denotes a standard Brownian motion. I assume that investors do not observe the drift $\theta(t)$. They only know that it can be any of n possible values $\theta_1 < \theta_2 < \cdots < \theta_n$ and that in any infinitesimal time-interval Δ there is probability $p\Delta$ that a new drift will be chosen according to the probability distribution $f = (f_1,\ldots,f_n)$. Since there are no restrictions on n, we can think of the points in $\Theta = \{\theta_1,\ldots,\theta_n\}$ as forming a fine grid on the real interval $[\theta_1,\theta_n]$.

Even though investors do not observe the true drift, I assume they observe a noisy signal:

$$de = \theta dt + \sigma_e dB_e,$$

where B_e is a standard Brownian motion independent of B_D. This form of the signal is the continuous time analog of the standard "signal equals funda-

[1]Dividend and output are used interchangeably throughout the paper. Since, in equilibrium, output also equals consumption, the three words are actually synonyms in this setup.

mentals plus noise;" that is, $e_t = \theta_t + \varepsilon_t$ with ε_t normally distributed, in a discrete time model (e.g., see Detemple (1986)). The inverse of the diffusion parameter,

$$h_e = 1/\sigma_e,$$

reflects the *precision* of the external signal. I say that investors have precise signals when h_e is relatively high. When $p = 0$ and h_e approaches infinity, the model gets closer to the standard textbook model where investors know the constant drift rate θ. Similarly, the precision of the "dividend signal" is

$$h_D = 1/\sigma_D.$$

The main goal of this paper is to characterize asset returns in this economy with parameter uncertainty and study their behavior for different values of the precision of the external signal h_e. Equilibrium prices and interest rates are determined in equilibrium by standard market clearing conditions. Specifically, if P denotes the price of the risky asset and r the instantaneous interest rate, then investors choose the fraction of wealth invested in stock, $\alpha(t)$, and consumption, $c(t)$, in order to solve the maximization problem:

$$\max_{c,\alpha} E\left[\int_0^\infty u(c,s)ds \,|\, \mathcal{F}(0)\right], \tag{1}$$

subject to

$$dW = W\left[\alpha\left(\frac{dP + Ddt}{P}\right) + (1-\alpha)rdt\right] - cdt. \tag{2}$$

An *equilibrium* is defined by a vector of processes $(c(t), \alpha(t), P(t), r(t))$ such that the maximization problem is solved and markets clear. That is, $\alpha(t) = 1$ and $c(t) = D(t)$.

II. The Dynamics of Investors' Beliefs

Let me denote investors' information set at time t by $\mathcal{F}(t)$. This contains all past realizations of dividends and signals. Let $\pi_i(t)$ be investors' beliefs that the drift rate is θ_i at time t, conditional on their information $\mathcal{F}(t)$:

$$\pi_i(t) = Prob(\theta(t) = \theta_i \,|\, \mathcal{F}(t)). \tag{3}$$

Also, let me denote the vector of these probabilities by $\Pi = (\pi_1, \ldots, \pi_n)$: This distribution summarizes investors' overall information at time t. Given these beliefs, they can compute the expected drift rate at time t:

$$m_\theta \equiv E(\theta \,|\, \mathcal{F}(t)) = \sum_{i=1}^{n} \pi_i \theta_i. \tag{4}$$

The following lemma shows that the evolution over time of investors' beliefs $\pi_i(t)$ can be described as a diffusion process.

LEMMA 1: *(a) Suppose that at $t = 0$ investors' beliefs are represented by the prior probability distribution $(\hat{\pi}_1, \ldots, \hat{\pi}_n)$. Then, for all $i = 1, \ldots, n$:*

$$d\pi_i = p(f_i - \pi_i)dt + \pi_i(\theta_i - m_\theta)(h_D d\tilde{B}_D + h_e d\tilde{B}_e) \quad (5)$$

for $t \geq 0$ subject to the initial condition $\pi_i(0) = \hat{\pi}_i$ for all $i = 1, \ldots, n$. In this equation

$$d\tilde{B}_D = h_D\left(\frac{dD}{D} - m_\theta dt\right)$$

$$d\tilde{B}_e = h_e(de - m_\theta dt)$$

are standard Brownian motions with respect to the information filtration $\mathcal{F}(t)$.

(b) *For all $i = 1, \ldots, n$, if $\pi_i(0) > 0$ then for every finite t,*

$$Prob(\pi_i(t) > 0) = 1.$$

Expression 5 is quite intuitive: The stochastic components $d\tilde{B}_D$ and $d\tilde{B}_e$ are the normalized innovation processes of dividend and signal realizations. Since each of them enters in equation (5) normalized by its own precision parameter, signals have greater weight in investors' posterior distribution than dividends whenever they have higher precision—that is, whenever $h_e > h_D$. The drift $p(f_i - \pi_i)$ is a mean-reverting component that pulls π_i toward f_i, which is the relative proportion of time that $\theta(t)$ equals θ_i in the long run. It is intuitive that, other things equal, a higher frequency p of shifts implies that the conditional distribution $(\pi_1, \ldots, \pi_n)$ is "closer" to the unconditional one $(f_1, \ldots, f_n)$. Hence, the speed of mean reversion in equation (5) is given by p.[2]

In order to gather some more intuition about the process of equation (5), it is useful to rewrite it in terms of the original processes B_D and B_e. This exercise yields the description of the process $d\pi_i$ from the perspective of an outside observer who knows that during some time interval $[t_1, t_2]$ the true drift rate $\theta(t)$ is equal to a particular θ_ℓ. As the following corollary shows, we can then gauge how the precision of the signals affects the dispersion of investors' beliefs around the true state.

[2] Notice also that every solution $(\pi_1(t), \ldots, \pi_n(t))$ to equation (5) has the property that for all $t \geq 0$, $\sum_{i=1}^n \pi_i(t) = 1$. In fact, from Ito's lemma it can be immediately verified that the quantity $S = \sum_{i=1}^n \pi_i$ is such that $dS = 0$ for all t. See Liptser and Shiryayev (1977) for details.

COROLLARY 1: *Suppose that the conditions of Lemma 1 are satisfied and let the true state be* $\theta(t) = \theta_\ell$ *for* $t \in [t_1, t_2]$. *Then, for* $t_1 < t < t_2$ *and for all* $i = 1, \ldots, n$:

$$\begin{aligned} d\pi_i &= [p(f_i - \pi_i) + k\pi_i(\theta_i - m_\theta)(\theta_\ell - m_\theta)]dt \\ &\quad + \pi_i(\theta_i - m_\theta)(h_D dB_D + h_e dB_e), \end{aligned} \tag{6}$$

where

$$k = h_D^2 + h_e^2.$$

Expression (6) shows that when θ_ℓ is the actual drift rate of the observable processes dD and de during some period of time $[t_1, t_2]$, the drift of $d\pi_i$ has a second component $k\pi_i(\theta_i - m_\theta)(\theta_\ell - m_\theta)$ which tends to pull π_i toward one if $i = \ell$ and toward zero if $i \neq \ell$. In fact, notice that for $i = \ell$ this second component equals $k\pi_\ell(\theta_\ell - m_\theta)^2 > 0$, and hence it has the effect of increasing π_ℓ over time. However, as π_ℓ gets closer to one, the term $(\theta_\ell - m_\theta)$ converges to zero and so do both $k\pi_\ell(\theta_\ell - m_\theta)^2$ and the diffusion term in equation (6). Hence, eventually the first component in the drift $p(f_\ell - \pi_\ell)$ would dominate, preventing π_ℓ from converging to one. Moreover, the speed at which π_ℓ is attracted to one is given by the constant k, which in turn depends on the precision of the signal h_e: Higher precision implies faster learning. Of course, for $i \neq \ell$ the probability π_i tends to converge to zero.

In summary, the specific intertemporal behavior of investors' beliefs depends on the relative sizes of the parameters p and k. For given p the distribution Π tends to be more concentrated when signals are more precise (higher k), for given k the distribution tends to be closer to the stationary $f = (f_1, \ldots, f_n)$ when p is higher. Notice finally that part (b) of Lemma 1 implies that if investors give a positive probability to each state θ_i at time zero, then $\pi_i(t) > 0$ for all $i = 1, \ldots, n$ and all t. I will assume throughout that $\pi_i(t) > 0$ for all i and t.

III. Asset Prices, Excess Returns, and Investors' Uncertainty

In this section I obtain formulas for the equilibrium stock price and interest rate and then study the behavior of stock market returns.

PROPOSITION 1: (a) *The equilibrium price function* $P(\Pi, D)$ *is*

$$P(\Pi, D) = D\left(\sum_{i=1}^{n} \pi_i C_i\right), \tag{7}$$

where C_i *are positive constants characterized by*

$$C_i = \frac{1}{u_c(D(t), t)D(t)} E\left(\int_t^\infty u_c(D(s), s)D(s)ds \,|\, \theta(t) = \theta_i\right).$$

(b) *The equilibrium interest rate r is:*

$$r = \phi + \gamma m_\theta - \tfrac{1}{2}\gamma(\gamma + 1)\sigma_D^2. \tag{8}$$

Each constant C_i represents the investors' expectation of future dividends conditional on the state being θ_i today, discounted by the marginal utility of consumption and normalized to make it independent of the current dividend and time. Hence, a high C_i implies that investors would be willing to pay a high price relative to the current dividend in state θ_i. Since they do not actually observe the state θ_i, they weight each C_i by its conditional probability π_i, thereby obtaining equation (7).

The next proposition characterizes equilibrium stock returns. For notational convenience, let me denote the total *excess* returns by

$$dR = \frac{dP + Ddt}{P} - rdt. \tag{9}$$

PROPOSITION 2: *The equilibrium excess returns follow the process:*

$$dR = \mu_R dt + (\sigma_D + h_D V_\theta) d\tilde{B}_D + V_\theta h_e d\tilde{B}_e, \tag{10}$$

where

$$\mu_R = \gamma(\sigma_D^2 + V_\theta) \tag{11}$$

$$V_\theta = \frac{\sum_{i=1}^{n} \pi_i C_i(\theta_i - m_\theta)}{\sum_{i=1}^{n} \pi_i C_i}. \tag{12}$$

Notice both the drift and the volatility of equilibrium stock returns can be fully characterized by studying the behavior of a single quantity V_θ. This task is undertaken below.

A. Investors' Uncertainty

In this section I discuss the behavior of the quantity V_θ that characterizes the stock return process given in equation (10). Let me define the following adjusted distribution on the state space $\Theta = (\theta_1, \ldots, \theta_n)$:

$$\pi_i^* = \frac{\pi_i C_i}{\sum_{j=1}^{n} \pi_j C_j} \tag{13}$$

The distribution $\Pi^* = (\pi_1^*, \ldots, \pi_n^*)$ is adjusted to reflect the investors' marginal valuation of the risky asset in the various states. I call this distribution the *value-adjusted* distribution. Let

$$m_\theta^* = E^*(\theta|\mathcal{F}(t)) = \sum_{i=1}^{n} \pi_i^* \theta_i \tag{14}$$

be the expected growth rate of the economy according to the value-adjusted distribution. It is immediate to see that

$$V_\theta = m_\theta^* - m_\theta \tag{15}$$

Therefore, V_θ reflects the relative distance between the true expected growth rate of the economy and the value-adjusted expected growth rate. Intuitively, V_θ can be considered a summary of both investors' degree of uncertainty about the true growth rate θ as well as the impact of this uncertainty on the investors' own valuation of the asset. For example, when Π is a degenerate distribution giving probability one to a state θ_ℓ or when C_i is constant for all i, then V_θ is zero. In the first case there is no uncertainty; in the second case uncertainty does not matter because investors assign the same value to the asset in every state. However, in general V_θ is different from zero; in fact, V_θ tends to be bigger (in absolute value) either when investors have more diffuse beliefs or when they value the asset very differently across states.

Finally, the sign of V_θ is also important: If V_θ is positive, then on average investors deem the asset more valuable in states that have higher growth rate than m_θ, whereas when V_θ is negative they deem the asset more valuable in states with a lower growth rate.

The following two lemmas make the above statements formal: The first characterizes the vector $\mathbf{C}$ and the second the quantity V_θ.

LEMMA 2: *Define the constant*

$$K = \sum_{i=1}^{n} \frac{f_i}{\phi + p + (\gamma - 1)\theta_i + \frac{1}{2}\gamma(1-\gamma)\sigma_D^2} \tag{16}$$

and let $C(\theta)$ be the continuous function on the interval $[\theta_1, \theta_n]$ defined as:

$$C(\theta) = \frac{1}{(\phi + p + (\gamma - 1)\theta + \frac{1}{2}\gamma(1-\gamma)\sigma_D^2)(1 - pK)}. \tag{17}$$

Then, for all $i = 1, \ldots, n$, we have $C_i = C(\theta_i)$.

The function $C(\theta)$ is monotonic and convex, and it is *decreasing* in θ if and only if $\gamma > 1$. That is, investors who are more risk averse than the log-utility investor assign a lower relative value to the asset in higher growth rate states. From the definition of C_i in Proposition 1 this is not surprising; in fact, investors discount future dividends using their marginal utility of future consumption. Since in equilibrium consumption equals dividends, even though a higher growth rate implies a higher expectation of future consumption, investors' discount rates are also higher. It is easy to see that the effect on the discount rate dominates when $\gamma > 1$. In other words, when investors with high coefficients of risk aversion expect *low* consumption growth, their hedging demand for assets increases. Since the supply of the risky asset is fixed while the riskless asset is in zero-net supply, this demand for assets decreases the interest rate and increases the price of the risky asset relative to dividends. This implies that $C(\theta)$ is decreasing. Figure 1 plots the function $C(\theta)$ for various values of γ.

LEMMA 3: *V_θ can be characterized as follows:*

(a) *$V_\theta < 0$ if and only if $\gamma > 1$. $V_\theta = 0$ if and only if $\gamma = 1$.*
(b) *Let $C(\theta)$ be nonconstant and let $\tilde{\Pi}$ be a mean-preserving spread of Π.*[3] *Then $\tilde{V}_\theta < V_\theta$ if $\gamma > 1$ and $\tilde{V}_\theta > V_\theta$ if $\gamma < 1$, where "$\sim$" denotes a quantity computed using the distribution $\tilde{\Pi}$.*
(c) *V_θ decreases as γ increases.*

Part (a) of Lemma 3 shows that if investors have high risk aversion, then the value-adjusted distribution gives more weight to the low-growth states. This result stems immediately from the fact that $C(\theta)$ is decreasing and convex for $\gamma > 1$.

Part (b) of the lemma instead shows that an increase in "uncertainty" on the growth rate of the economy increases V_θ in absolute value. The intuition is straightforward as well: A mean-preserving spread increases the dispersion of the distribution Π and hence the relative weight given to the tails of the distribution. Since the function $C(\theta)$ is convex, the value-adjusted probability distribution becomes even more skewed toward the high-value states. This increases the effect of the value adjustment, thus the absolute distance between m_θ^* and m_θ increases.

Part (c) relates the measure V_θ to the preference parameter γ. The reason that it holds can be grasped from Figure 1: When $\gamma < 1$, an increase in γ makes $C(\theta)$ less convex and hence the effect of the value adjustment on the mean growth rate m_θ^* decreases. Hence, $V_\theta = m_\theta^* - m_\theta$ decreases as we increase γ toward one. When $\gamma > 1$, an increase in γ increases the convexity of

[3] A mean-preserving spread of the distribution Π is given by $\tilde{\Pi}$ defined by $\tilde{\pi}_i = \pi_i + s_i$ where for $i_1 < i_2 < i_3 < i_4$, $s_{i_1} = -s_{i_2} = \alpha > 0$, $s_{i_4} = -s_{i_3} = \beta > 0$, $s_i = 0$ otherwise, and such that $1 > \tilde{\pi}_i > 0$ and $\alpha(\theta_{i_1} - \theta_{i_2}) = \beta(\theta_{i_3} - \theta_{i_4})$. Intuitively, a mean-preserving spread moves probability mass from the "center" of the distribution toward its "tails" without changing the mean (see, e.g., Ingersoll (1987)).

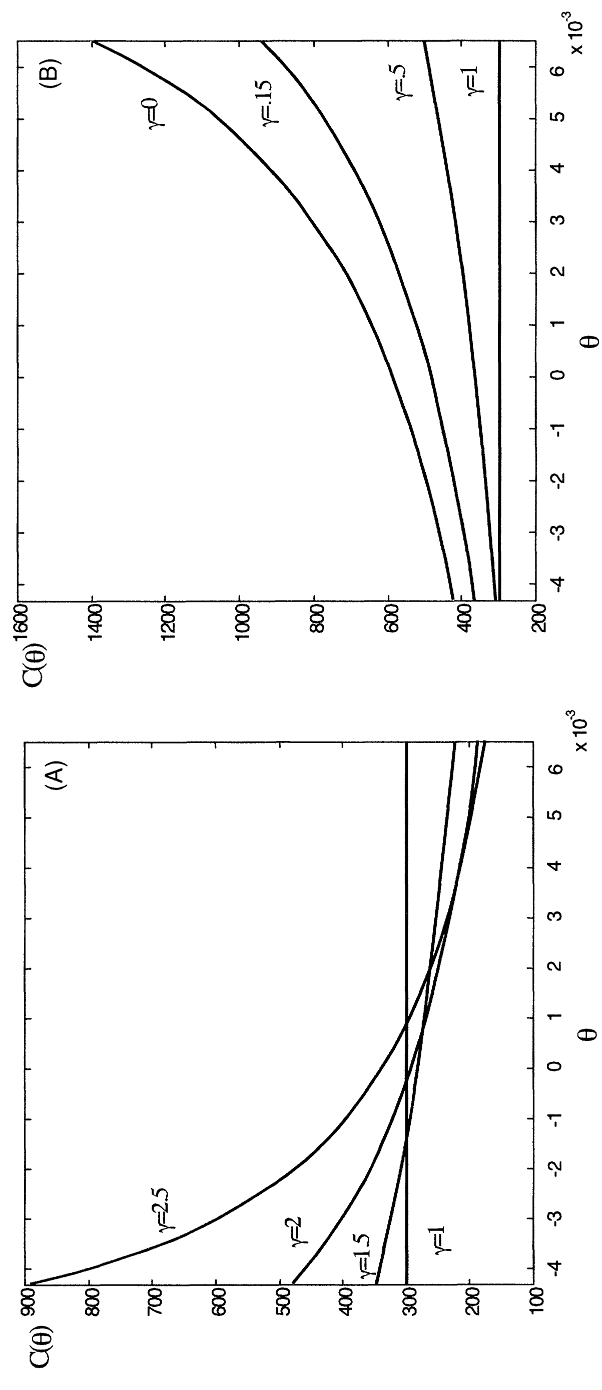

Figure 1. The function $C(\theta)$. (A) plots the function $C(\theta)$ for various values of investors' coefficient of risk aversion $\gamma \geq 1$. This function represents investors' marginal valuation of the stock as a multiple of the current dividend when they condition on the true drift of the dividends process being θ. (B) plots the same function for values of $\gamma \leq 1$.

$C(\theta)$, thereby increasing the effect of the "value adjustment" on the mean growth rate m^*_θ. This increases V_θ in absolute value. Since V_θ is negative for $\gamma > 1$, this implies that V_θ decreases further as we increase γ.

B. Investors' Uncertainty and Expected Stock Returns

This section discusses the features of investors' expected returns μ_R. I start with a formal characterization in the following proposition.

PROPOSITION 3:

(a) *If $\gamma > 1$, then higher uncertainty decreases the risk premium. That is, a mean-preserving spread on investors' beliefs Π decreases μ_R.*
(b) *If either $m_\theta > \sigma_D^2 + \theta_1$ or $\pi_1 < \bar{\pi}_1$ where $\bar{\pi}_1$ is given in equation (A35) of the Appendix, the expected excess return μ_R decreases with γ for γ sufficiently high. Hence, μ_R is bounded above.*
(c) *If $m_\theta > \sigma_D^2 + \theta_1$, there is $\bar{\gamma}$ such that $\mu_R < 0$ for $\gamma > \bar{\gamma}$. Moreover, a mean-preserving spread on Π decreases $\bar{\gamma}$.*

Part (a) of Proposition 3 shows that there is no premium for uncertainty. Actually, quite the opposite holds. From the characterization of the probability distribution in Lemma 1 a low precision of signal h_e implies that the posterior distribution Π tends to be more diffuse on the space Θ. Hence, Proposition 3(a) implies that when public signals are less precise the expected excess return is smaller. In other words, when there is "better" information about the state of the economy, there is also a relatively high risk premium. As explained in the introduction, the intuitive explanation of this seemingly paradoxical result stems from the standard result that the risk premium depends on the covariance of consumption growth and stock returns; that is,

$$\mu_R = \gamma(\sigma_D^2 + V_\theta) = \gamma \times Cov\left(dR, \frac{dc}{c}\right). \tag{18}$$

When the external signal is very precise, the covariance between consumption and stock returns is higher than in the case where signals are less precise. In fact, in the latter case a negative innovation in dividends has the direct effect of decreasing the price of the stock and the indirect effect of increasing investors' hedging demand for the asset because they now expect lower consumption in the future. This indirect positive effect on the price partly dampens the negative direct effect due to the decrease in dividends. However, as we increase the precision of the signal h_e, investors' hedging demand is less and less affected by dividend realizations because investors' expectations depend more and more on the signal. Since in equilibrium dividends equal consumption, the above discussion entails that as we increase the precision of external signals the covariance of returns and consumption increases and so does the risk premium.

Part (b) of this proposition shows that when either investors' expected growth rate is not too low or π_1 is not too high, there is an upper bound to the risk premium. The intuition for this result is related to that of part (a): As usual, for given positive covariance of returns and consumption growth, higher risk aversion implies a higher risk premium. When signals are not precise though, an increase in risk aversion also implies a bigger impact of dividend realizations on investors' hedging demand. As we saw above, this decreases the covariance itself. Moreover, the second effect dominates for very high γ so that μ_R decreases as γ increases. I should point out that the conditions of Proposition 3(b) are generally satisfied when signals are not precise and n is large, so that the distribution Π is diffuse. In fact, for typical parameter values of ϕ, p, and σ_D^2, $\bar{\pi}_1$ is extremely high, well above 0.85.

Finally, part (c) shows that when investors' expected growth rate is not too low, the risk premium turns negative when investors are sufficiently risk averse. Once again, this result stems from the fact that high expected future dividends tend to decrease the covariance between current returns and consumption growth. For sufficiently high γ this covariance becomes negative (see Campbell (1999) for a discussion of a related point).

Figure 2, Panel A, plots μ_R against the standard deviation of investors' beliefs $\sigma_\theta = \sqrt{\sum_{i=1}^{n} \pi_i(\theta_i - m_\theta)^2}$ for various coefficients of risk aversion.[4] As σ_θ increases, μ_R decreases and becomes negative for a high γ. Similarly, Figure 2, Panel B, plots μ_R against the coefficient of risk aversion γ for $\sigma_\theta = 0.11\%$. To better understand the effect of changing uncertainty over time, Figure 3, Panel A, plots μ_R for various values of γ resulting from one simulation of dividends and posterior distributions. We can see that a high γ does not imply a high μ_R, which at times can turn negative. From Figure 3, Panel B, we also notice that μ_R decreases when σ_θ increases.

C. Investors' Uncertainty and the Conditional Volatility of Stock Returns

From the equilibrium process for returns (equation (10)) it is immediately evident that the quantity V_θ characterizes return volatility as well. In fact,

$$\sigma_R^2 = \sigma_D^2 + V_\theta[2 + (h_e^2 + h_D^2)V_\theta]. \tag{19}$$

The following proposition then holds.

PROPOSITION 4:

(a) *σ_R is a U-shaped function of γ with $\sigma_R = \sigma_D$ for $\gamma = 1$. Moreover, a mean-preserving spread on Π increases σ_R if $\sigma_R > \sigma_D$. The effect is ambiguous if $\sigma_R < \sigma_D$.*

(b) *Under the conditions of Proposition 3(c), if $h_e > h_D$ then $\sigma_R > \sigma_D$ for a sufficiently high coefficient of risk aversion.*

[4] The following parameters are used (in monthly units): $\sigma_D = 1.5\%$, $p = 1.67\%$, $\sigma_e = \infty$ (no external signal), and $\sigma_\theta = 0.11\%$ for Figure 2, Panel B.

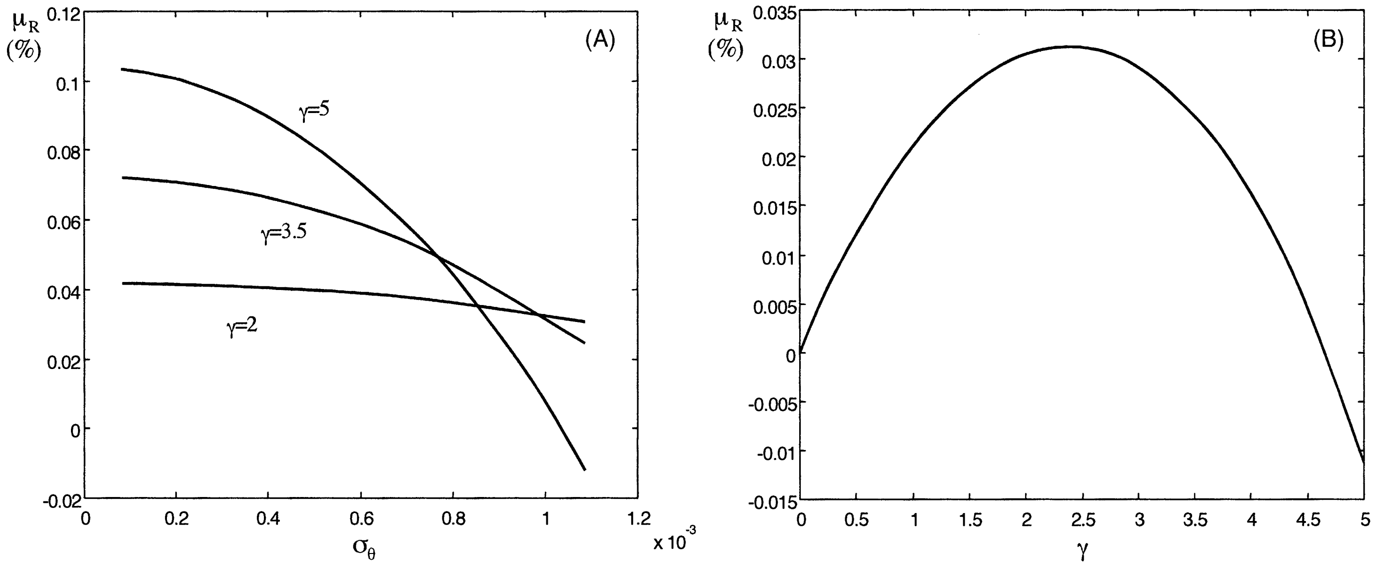

Figure 2. Expected returns and investors' uncertainty. (A) plots the conditional risk premium μ_R against the standard deviation of investors' beliefs $\sigma_\theta = \sqrt{\sum_{i=1}^{n} \pi_i(\theta_i - m_\theta)}$, which proxies for investors' uncertainty, for various coefficients of risk aversion. (B) plots the conditional risk premium μ_R against the coefficient of risk aversion γ for a level of $\sigma_\theta = 0.11\%$.

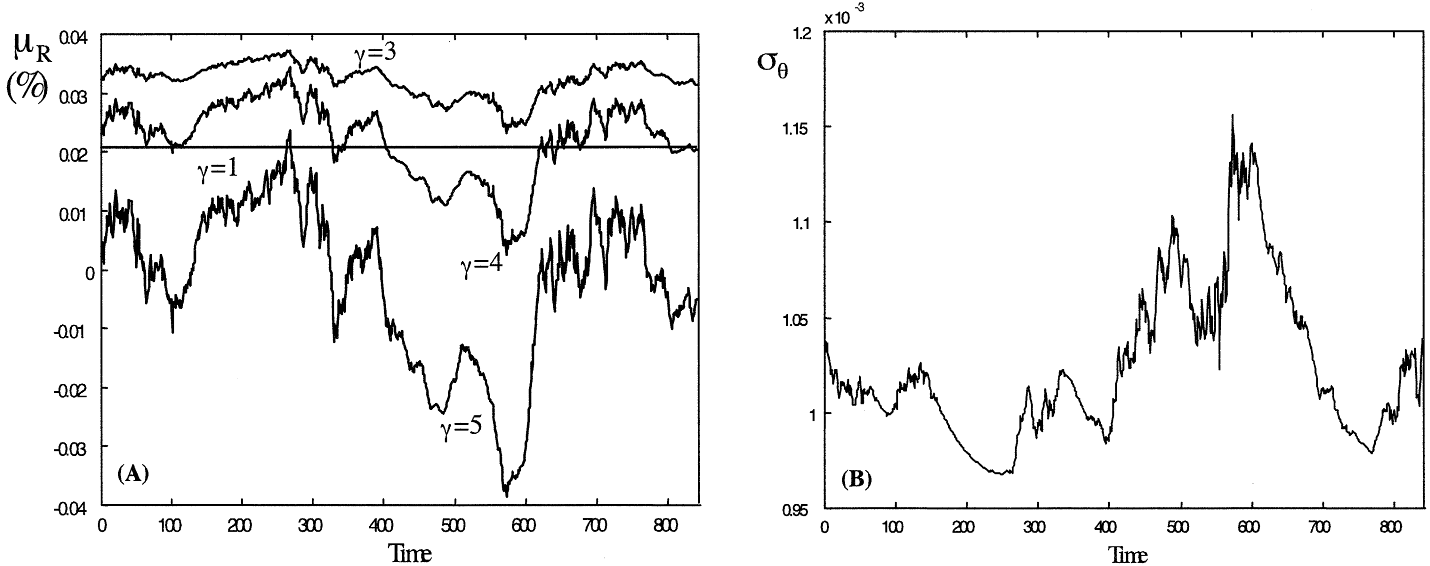

Figure 3. Expected returns and investors' uncertainty over time. (A) plots the conditional risk premium μ_R over time as the result of one Monte Carlo simulation of dividends and posterior distributions. μ_R is computed for the coefficients of risk aversion $\gamma = 1,3,4,5$. (B) plots the standard deviation of investors' beliefs σ_θ across time.

The intuition behind the results in Proposition 4 is the same as the one for the risk premium. For logarithmic investors, uncertainty (and hence the precision of signals) does not matter because they value the asset the same independently of the state. Hence $V_\theta = 0$ and $\sigma_R = \sigma_D$. When there is uncertainty about the true drift (Π nondegenerate), if $\gamma > 1$ any positive innovation in dividends increases the price but decreases investors' hedging demand due to the increase in investors' expectations of future consumption. Hence, the effect on the price of dividend realizations is not as strong and return volatility decreases. However, for very high levels of risk aversion the effect on the hedging demand outweighs the direct effect on the price, thereby increasing volatility again. In contrast, for $\gamma < 1$ a positive realization of dividends increases investors' demand for the asset (substitution effects dominate), thereby further increasing the price of the stock. Hence $\sigma_R > \sigma_D$ if $\gamma < 1$.

D. The Risk Premium and the Conditional Variance of Returns

From the results about risk premium and return volatility, it is clear that the relationship between return volatility and expected returns is ambiguous and depends on the degree of investors' uncertainty. This statement can be made precise by noticing that from equations (11) and (19) we can write

$$\mu_R = \gamma\sigma_R^2 - \gamma V_\theta[1 + (h_e^2 + h_D^2)V_\theta]. \tag{20}$$

It is apparent then that the relationship between the conditional risk premium and the conditional variance of returns is linear but investors' uncertainty biases this relationship through V_θ in an ambiguous way. In fact, the second term in equation (20) can be positive or negative depending on the magnitude of V_θ. Specifically, for log-utility or when signals are very precise, V_θ is approximately zero and hence a linear positive relationship results. In contrast, when $\gamma > 1$ and signals are not precise, the second term in equation (20) is positive for $-1/(h_e^2 + h_D^2) < V_\theta < 0$ and negative for $V_\theta < -1/(h_e^2 + h_D^2)$. Since the magnitude of V_θ changes over time due to investors' fluctuating level of uncertainty, equation (20) implies that there is no precise relationship between expected excess returns and conditional volatility. Indeed, the empirical finance literature has long documented that the evidence for a positive relationship between expected returns and conditional return variance is very weak at best (e.g., see Campbell et al. (1999) and Scruggs (1998)).

IV. Conclusions

This paper shows that the relationship between the precision of public information about economic growth and the performance of the stock market is nontrivial. In a standard Lucas economy where the growth rate of output is unknown but where investors receive signals about it, I obtain results on equity premium and return volatility that can be deemed counterintuitive at first: (i) More precise signals on the true state of the economy—that is, better information—tend to increase the equity premium. Therefore, poor information does not demand a risk premium. (ii) If information is imprecise, then an increase in the risk aversion coefficient does not necessarily increase the equity risk premium. In fact, there is an *upper bound* to the equity risk premium. Moreover, higher uncertainty (i.e., poorer signals) implies that the upper bound is achieved for a lower coefficient of risk aversion γ. (iii) When signals are imprecise, return volatility is U-shaped with respect to investors' coefficient of risk aversion. (iv) The relationship between expected returns and return volatility is ambiguous and depends on investors' level of uncertainty.

The channel through which the precision of public information affects stock returns is its influence on the equilibrium covariance between current consumption and returns. In fact, there is a qualitative difference between dividends and "other statistics" as signals of future economic performance: Though "other statistics" affect only investors' expectations, dividend realizations also affect investors' consumption possibilities. Hence, the implied covariance between consumption and return is modified as we change the precision of external signals.

A few final remarks are in order. First, the model implies that when the external signals are not precise the conditional expected excess return may become negative when the coefficient of risk aversion is high. This occurs when investors' uncertainty increases. The same factor also decreases the dividend yield, thereby generating a positive relation between dividend yield and expected returns. A very low dividend yield may be associated with a negative risk premium (e.g., see Lamont (1998)).

Second, the empirical literature has had a hard time determining the relationship between expected returns and their conditional variance (see, e.g., Campbell et al. (1999) and Scruggs (1998)). Result (iv) justifies this finding: Investors' fluctuating uncertainty over time about the true growth rate of the economy makes this relationship ambiguous because it introduces a bias that is at times positive (for low level of uncertainty) and at times negative (for high level of uncertainty).

Third, the model developed in this paper assumes that investors have a power utility function. This choice enables me to obtain a simple closed-form solution for asset prices but it imposes also a specific relationship between investors' degree of relative risk aversion and their elasticity of intertemporal substitution, the latter being the reciprocal of the former. This strict relationship makes it more difficult to interpret exactly the comparative statics results obtained in this paper. Nonetheless, the basic intuition developed in the model is likely to remain even under a more general utility function. In fact, an increase in risk aversion would still imply that investors' hedging demand for assets increases after bad news in dividends, thereby counterbalancing the negative pressure in prices due to the negative dividend news. With a power utility function this effect is strong because a higher risk aversion implies a lower elasticity of intertemporal substitution, which in turn also increases the demand for the assets after negative shocks in dividends. In fact, negative dividend innovations signal lower future consumption, which in turn implies higher savings to smooth out consumption. The exact balance between the two effects can only be assessed by using a more general utility function, such as Epstein and Zin (1989); this investigation is an interesting topic for future research.

Finally, this paper investigates a particular type of quality of information: information about economic growth. There are other types of information that are certainly relevant and that are also worth investigating. These may include information about future volatilities or correlations for example. The effect of "information quality" on these variables may have different implications on stock returns than the one discussed here. Also, it would be interesting to study how information quality about different firms' growth prospects affects their stock returns. The relationship between the cross section of returns and investors' information has been already addressed in the Bayesian CAPM literature (see, e.g., Barry and Brown (1985)). However, a model of intertemporal learning is still missing. One of the effects that we can reasonably expect is that equilibrium "betas" would tend to change over time as uncertainty fluctuates and investors' change their hedging demand for stocks.

Appendix

Proof of Lemma 1: (a) This is a slight generalization to the vector case of Theorem 9.1 in Liptser and Shiryayev (1977, p. 333). Let $\mathbf{X}(t)$ be an N-dimensional Ito process described by

$$d\mathbf{X} = \mu dt + \Sigma d\mathbf{W},$$

where $\mathbf{W}(t)$ is an M-dimensional Brownian motion. It is assumed that the N-dimensional vector $\boldsymbol{\mu}(t)$ follows an n state continuous-time Markov chain, where

$$\Lambda = \begin{pmatrix} -\sum_{j\neq 1} \lambda_{1j} & \lambda_{12} & \lambda_{13} & \cdots & \lambda_{1n} \\ \lambda_{21} & -\sum_{j\neq 2} \lambda_{2j} & \lambda_{23} & \cdots & \lambda_{2n} \\ \lambda_{31} & \lambda_{32} & -\sum_{j\neq 3} \lambda_{3j} & \cdots & \lambda_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_{n1} & \lambda_{n2} & \lambda_{n3} & \cdots & -\sum_{j\neq n} \lambda_{nj} \end{pmatrix}$$

is the infinitesimal matrix. Notice that for all $i = 1,\ldots,n$, $\lambda_{ii} = -\sum_{j\neq i}\lambda_{ij}$ (e.g., see Karlin and Taylor (1975), p. 151). Both $\boldsymbol{\mu}$ and Σ can be functions of $\mathbf{X}$. Then, the proposition of Theorem 9.1 in Liptser and Shiryayev becomes:

THEOREM: *For given prior distribution $(\hat{\pi}_1,\ldots,\hat{\pi}_n)$ on $(\mu_1,\ldots,\mu_n)$, under some technical conditions (see Liptser and Shiryayev (1977)), the posterior probability*

$$\pi_i(t) = Prob(\boldsymbol{\mu}(t) = \boldsymbol{\mu}_i | \mathcal{F}(t))$$

satisfies the system of stochastic differential equations

$$d\pi_i(t) = \sum_{j=1}^{n} \lambda_{ji}\pi_j(t)dt + \pi_i(t)(\boldsymbol{\mu}_i - \bar{\boldsymbol{\mu}})'(\Sigma\Sigma')^{-1/2}d\widetilde{\mathbf{W}} \tag{A1}$$

under the condition $\pi_i(0) = \hat{\pi}_i$ where

$$\bar{\boldsymbol{\mu}} = \sum_{j=1}^{n} \boldsymbol{\mu}_j \pi_j$$

$$d\widetilde{\mathbf{W}} = (\Sigma\Sigma')^{-1/2}(d\mathbf{X} - \bar{\mu}dt).$$

The proof of this claim is identical to that of Liptser and Shiryayev's Theorem 9.1 after the obvious changes are made. Similarly, Liptser and Shiryayev's Theorem 9.2 shows that under some technical conditions equation (A1) admits a unique nonnegative strong solution.

Finally, in order to obtain equation (5), we define $\mathbf{X} = (D,e)'$, $\boldsymbol{\mu}_i = (\theta_i D, \theta_i)'$, $\Sigma_{11} = D\sigma_D$, $\Sigma_{22} = \sigma_e$, $\Sigma_{ij} = 0$ for $i \neq j$, $\mathbf{W} = (B_D, B_e)'$, $\lambda_{ij} = pf_j$ for $i \neq j$, and $\lambda_{ii} = -\sum_{j\neq i}\lambda_{ij} = -p\sum_{j\neq i} f_j = pf_i - p$. We then obtain

$$\begin{aligned} d\pi_i &= \left(\sum_{j=1}^{n} pf_i\pi_j - p\pi_i\right)dt + \pi_i\left\{(\theta_i - m_\theta)D\frac{1}{\sigma_D D}\left[\frac{1}{\sigma_D D}(dD - m_\theta D dt)\right]\right. \\ &\qquad \left. + (\theta_i - m_\theta)\frac{1}{\sigma_e}\left[\frac{1}{\sigma_e}(de - m_\theta dt)\right]\right\} \qquad \text{(A2)} \\ &= p(f_i - \pi_i)dt + \pi_i(\theta_i - m_\theta)(h_D d\tilde{B}_D + h_e d\tilde{B}_e). \end{aligned}$$

Finally, using the fact that $d\tilde{B}_D = dB_D + h_D(\theta(t) - m_\theta)dt$ and $d\tilde{B}_e = dB_e + h_e(\theta(t) - m_\theta)dt$, substitution in the above formulas yields equation (6).

(b) See Lemma 9.3 in Liptser and Shiryayev (1977, p. 342). Q.E.D.

Proof of Proposition 1: (a) From the first-order conditions on investors' portfolio problem we obtain the standard formula

$$P(t)\rho(t) = E_t\left[\int_t^\infty \rho(s)D(s)ds\right], \tag{A3}$$

where $\rho(t) = u_c(c(t),t) = e^{-\phi t}c(t)^{-\gamma}$ is the stochastic discount factor. In a Lucas economy, in equilibrium the market for consumption goods must clear so that for every τ we must have $c(\tau) = D(\tau)$. Hence, we can determine the asset pricing formula by substituting this equilibrium condition to obtain

$$\frac{P(t)}{D(t)} = E\left[\int_t^\infty e^{-\phi(s-t)}\left(\frac{D(s)}{D(t)}\right)^{1-\gamma} ds \,|\, \mathcal{F}(t)\right]. \tag{A4}$$

Consider first the following conditional expectation:

$$V(t,\theta_i) = E\left[\int_t^\infty e^{-\phi(s-t)}\left(\frac{D(s)}{D(t)}\right)^{1-\gamma} ds \,|\, \theta(t) = \theta_i\right]. \tag{A5}$$

From the assumption on the dividend process we have

$$\frac{D(s)}{D(t)} = \exp\left(\int_t^s \theta(u)\,du - \frac{1}{2}\sigma_D^2(s-t) + \sigma_D(B_D(s) - B_D(t))\right) \tag{A6}$$

for $s > t$. As usual, the process $\theta(t)$ is assumed right-continuous; that is, $\lim_{\Delta\to 0}\theta(t+\Delta) = \theta(t)$. Hence, we can consider an infinitesimal time interval Δ such that if $\theta(t) = \theta_i$ there is probability $o(\Delta)$ that a shift occurs before time $t + \Delta$ (we take the limit as $\Delta \to 0$ below). We can then write

$$\begin{aligned} V(t,\theta_i) &= E\left[\int_t^\infty e^{-\phi(s-t)}\left(\frac{D(s)}{D(t)}\right)^{1-\gamma} ds \,|\, \theta(t) = \theta_i\right] \\ &= E\left[\int_t^{t+\Delta} e^{-\phi(s-t)}\left(\frac{D(s)}{D(t)}\right)^{1-\gamma} ds \,|\, \theta(t) = \theta_i\right] \\ &\quad + E\left[\int_{t+\Delta}^\infty e^{-\phi(s-t)}\left(\frac{D(s)}{D(t)}\right)^{1-\gamma} ds \,|\, \theta(t) = \theta_i\right]. \end{aligned} \tag{A7}$$

Hence, since $\theta(t) = \theta_i$ during the infinitesimal interval Δ, we immediately find:

$$E\left[\int_t^{t+\Delta} e^{-\phi(s-t)}\left(\frac{D(s)}{D(t)}\right)^{1-\gamma} ds \,|\, \theta(t) = \theta_i\right] = \int_t^{t+\Delta} e^{\hat{\theta}_i(s-t)}ds = \frac{e^{\hat{\theta}_i\Delta} - 1}{\hat{\theta}_i}, \tag{A8}$$

where $\hat{\theta}_i = -\phi + (1-\gamma)\theta_i - \frac{1}{2}\sigma_D^2\gamma(1-\gamma)$. Similarly, since conditional on $\theta(t) = \theta_i$ during the infinitesimal interval Δ the random variables $D(t+\Delta)/D(t)$ and $D(s)/D(t+\Delta)$ are independent, the second expectation can be rewritten as

$$
\begin{aligned}
&E\left[\int_{t+\Delta}^{\infty} e^{-\phi(s-t)}\left(\frac{D(s)}{D(t)}\right)^{1-\gamma} ds \,|\, \theta(t) = \theta_i\right] \\
&= E\left[e^{-\phi\Delta}\left(\frac{D(t+\Delta)}{D(t)}\right)^{1-\gamma} \times \int_{t+\Delta}^{\infty} e^{-\phi(s-(t+\Delta))}\left(\frac{D(s)}{D(t+\Delta)}\right)^{1-\gamma} ds \,|\, \theta(t) = \theta_i\right] \\
&= E\left[e^{-\phi\Delta}\left(\frac{D(t+\Delta)}{D(t)}\right)^{1-\gamma} |\, \theta(t) = \theta_i\right] \\
&\quad \times E\left[\int_{t+\Delta}^{\infty} e^{-\phi(s-(t+\Delta))}\left(\frac{D(s)}{D(t+\Delta)}\right)^{1-\gamma} ds \,|\, \theta(t) = \theta_i\right] \\
&= e^{\hat{\theta}\Delta} \times E\left[\int_{t+\Delta}^{\infty} e^{-\phi(s-(t+\Delta))}\left(\frac{D(s)}{D(t+\Delta)}\right)^{1-\gamma} ds \,|\, \theta(t) = \theta_i\right] \\
&= e^{\hat{\theta}\Delta} \times \left[\left(1 - \sum_{j\neq i}\lambda_{ij}\Delta\right) E\left(\int_{t+\Delta}^{\infty} e^{-\phi s}\left(\frac{D(s)}{D(t+\Delta)}\right)^{1-\gamma} ds \,|\, \theta(t+\Delta) = \theta_i\right)\right. \\
&\quad \left. + \sum_{j\neq i}\lambda_{ij}\Delta E\left(\int_{t+\Delta}^{\infty} e^{-\phi s}\left(\frac{D(s)}{D(t+\Delta)}\right)^{1-\gamma} ds \,|\, \theta(t+\Delta) = \theta_j\right)\right] \\
&= e^{\hat{\theta}\Delta} \times \left[\left(1 - \sum_{j\neq i}\lambda_{ij}\Delta\right) V(t+\Delta, \theta_i) + \sum_{j\neq i}\lambda_{ij}\Delta V(t+\Delta, \theta_j)\right].
\end{aligned}
\tag{A9}
$$

Using a Taylor expansion, we can write $V(t+\Delta,\theta_i) = V(t,\theta_i) + V'(t,\theta_i)\Delta + o(\Delta)$. Hence, overall we obtain

$$
V(t,\theta_i) = \frac{e^{\hat{\theta}_i\Delta} - 1}{\hat{\theta}_i} + e^{\hat{\theta}_i\Delta}\left(V(t;\theta_i) + V'(t;\theta_i)\Delta - \sum_{j\neq i}\lambda_{ij}V(t;\theta_i)\Delta + \sum_{j\neq i}\lambda_{ij}\Delta V(t;\theta_j)\right).
$$

Rearranging and dividing by Δ, we have

$$
V(t;\theta_i)\left(\frac{1 - e^{\hat{\theta}_i\Delta}}{\Delta}\right) = \frac{e^{\hat{\theta}_i\Delta} - 1}{\hat{\theta}_i\Delta} + e^{\hat{\theta}_i\Delta}\left(V'(t;\theta_i) - \sum_{j\neq i}\lambda_{ij}V(t;\theta_i) + \sum_{j\neq i}\lambda_{ij}V(t;\theta_j)\right).
\tag{A10}
$$

By taking the limit as $\Delta \to 0$ and rearranging terms, we obtain a first-order, linear differential equation:

$$V'(t;\theta_i) = \left(\sum_{j \neq i} \lambda_{ij} - \hat{\theta}_i\right) V(t;\theta_i) - \sum_{j \neq i} \lambda_{ij} V(t;\theta_j) - 1. \tag{A11}$$

Let $V(t) = (V(t;\theta_1),\ldots,V(t;\theta_n))'$ and $\mathbf{A} = -\Lambda - (1-\gamma)diag(\Theta) + (\phi + \frac{1}{2}\gamma(1-\gamma)\sigma_D^2)\mathbf{I}_n$ so that

$$V'(t) = \mathbf{A}V(t) - \mathbf{1}_n.$$

We now realize that, by definition, $V(t;\theta_i)$ is *time homogeneous* for all i, which implies that $V'(t;\theta_i) = 0$ for all i. Hence, the solution is easily obtained by solving $\mathbf{0}_n = \mathbf{A}V(t) - \mathbf{1}_n$. That is,

$$V(t) = \mathbf{C} = \mathbf{A}^{-1}\mathbf{1}_n. \tag{A12}$$

Since by definition $P(t)/D(t) = \sum_{i=1}^{n} \pi_i V(t,\theta_i)$, we obtain the claim.

Finally, for the specification $\lambda_{ij} = pf_j$ for $j \neq i$, it is easy to see from $\mathbf{AC} = \mathbf{1}_n$ that for every i, we have

$$\left(\phi + p + (\gamma - 1)\theta_i + \frac{1}{2}(1-\gamma)\gamma\sigma_D^2\right)C_i = 1 + p\sum_{j=1}^{n} f_j C_j. \tag{A13}$$

Hence, since to have positive prices we need $C_i > 0$ for all is, the following condition must always be satisfied: For all $i = 1,\ldots,n$,

$$\phi + p + (\gamma - 1)\theta_i + \tfrac{1}{2}(1-\gamma)\gamma\sigma_D^2 > 0. \tag{A14}$$

Lemma 2 shows that under a slightly stricter condition than that given in equation (A14), the system of equations (A13) has a unique solution.

(b) Similar to case (a), it is possible to express the first-order conditions in differential form so that for every asset we have

$$0 = \rho D dt + E[d(\rho P)|\mathcal{F}(t)]. \tag{A15}$$

The risk-free asset can be thought of as an asset yielding an instantaneous dividend $D = rB$, where the price of the bond is always equal to 1. Hence, we obtain

$$rdt = -E_t\left[\frac{d\rho}{\rho}\right].$$

It is easy to see that

$$E_t\left[\frac{d\rho}{\rho}\right] = -\left(\phi + \gamma m_\theta - \frac{1}{2}\gamma(\gamma+1)\sigma_D^2\right)dt. \tag{A16}$$

Q.E.D.

Proof of Proposition 2: Notice first that from the definition of $d\tilde{B}_D$, I can rewrite the dividend process as $dD/D = m_\theta dt + \sigma_D d\tilde{B}_D$. Applying Ito's lemma to equation (7), we obtain

$$\begin{aligned} dP &= \sum_{i=1}^{n} \pi_i C_i dD + D\sum_{i=1}^{n} C_i d\pi_i + \sum_{i=1}^{n} C_i(d\pi_i dD) \\ &= P\mu_P dt + P\sigma_D d\tilde{B}_D + D\sum_{i=1}^{n} C_i \pi_i(\theta_i - m_\theta)(h_D d\tilde{B}_D + h_e d\tilde{B}_e) \end{aligned} \tag{A17}$$

with

$$\mu_P = \left(m_\theta + \frac{D}{P}p\sum_{i=1}^{n} C_i(f_i - \pi_i) + \frac{D}{P}\sum_{i=1}^{n} C_i\pi_i(\theta_i - m_\theta)\right).$$

Since from equation (7) $D = P/(\sum_{i=1}^{n} C_i\pi_i)$, the diffusion terms in equation (10) follow immediately. Next, for all $i = 1,\ldots,n$, multiply both the right- and left-hand sides of equation (A13) by π_i and then sum across $i = 1,\ldots,n$ to obtain the equality

$$\left(\phi + \frac{1}{2}(1-\gamma)\gamma\sigma_D^2\right)\sum_{i=1}^{n}\pi_i C_i + p\sum_{i=1}^{n}\pi_i C_i + (\gamma-1)\sum_{i=1}^{n}\pi_i\theta_i C_i = 1 + p\sum_{j=1}^{n} f_j C_j.$$

Manipulation of this expression yields

$$p\sum_{i=1}^{n} C_i(f_i - \pi_i) = \left(\phi + \frac{1}{2}(1-\gamma)\gamma\sigma_D^2\right)\sum_{i=1}^{n}\pi_i C_i + (\gamma-1)\sum_{i=1}^{n}\pi_i\theta_i C_i - 1.$$

Substituting this equality into μ_P along with $D/P = 1/(\sum_{i=1}^{n} C_i\pi_i)$ yields

$$\mu_P = \phi + \frac{1}{2}(1-\gamma)\gamma\sigma_D^2 + \gamma\frac{\sum_{i=1}^{n} C_i\pi_i\theta_i}{\sum_{i=1}^{n} C_i\pi_i} - \frac{1}{\sum_{i=1}^{n} C_i\pi_i}.$$

Finally, using equation (8) to write $\phi + \frac{1}{2}(1-\gamma)\gamma\sigma_D^2 = r - \gamma m_\theta + \gamma\sigma_D^2$, we obtain

$$E(dR) \equiv \mu_R \equiv \mu_P + \frac{D}{P} - r = \gamma\left(\sigma_D^2 + \frac{\sum_{i=1}^{n} C_i\pi_i(\theta_i - m_\theta)}{\sum_{i=1}^{n} C_i\pi_i}\right) = \gamma(\sigma_D^2 + V_\theta). \quad \text{(A18)}$$

Q.E.D.

Proof of Lemma 2: I prove the following claims:

(a) Let $\hat{\theta}_i = (\phi + (\gamma - 1)\theta_i + \frac{1}{2}(1-\gamma)\gamma\sigma_D^2)$ and $K = \sum_{i=1}^{n} f_i/(\hat{\theta}_i + p)$. If $K \neq 1/p$, then the constants

$$C_i = \frac{1}{(\hat{\theta}_i + p)(1 - pK)} \quad \text{(A19)}$$

are the unique solution to the system of equations (A13).

(b) $C_i > 0$ for all $i = 1,\ldots,n$ if and only if $K < 1/p$ and $\hat{\theta}_i + p > 0$ for all $i = 1,\ldots,n$.

(c) A sufficient condition for $K < 1/p$ and $\hat{\theta}_i + p > 0$ for all $i = 1,\ldots,n$ is the following: For all $i = 1,\ldots,n$,

$$\phi + (\gamma - 1)\theta_i + \tfrac{1}{2}(1-\gamma)\gamma\sigma_D^2 > 0. \quad \text{(A20)}$$

To prove claim (a) I first solve for the constants C_i, $i = 1,\ldots,n$, that solve the system of equations (A13). From equation (A13), we can rewrite C_i as

$$C_i = \frac{1}{\hat{\theta}_i + p} + \frac{p}{\hat{\theta}_i + p}\left(\sum_{j=1}^{n} f_j C_j\right). \quad \text{(A21)}$$

Multiply each side by f_i and sum across $i = 1,\ldots,n$ to obtain

$$\sum_{i=1}^{n} f_i C_i = \sum_{i=1}^{n} \frac{f_i}{\hat{\theta}_i + p} + p\left[\sum_{i=1}^{n} \frac{f_i}{\hat{\theta}_i + p}\right]\left(\sum_{j=1}^{n} f_j C_j\right). \quad \text{(A22)}$$

Taking the second term on the right-hand side to the left-hand side, and using the definition of K and the assumption that $K \neq 1/p$, we obtain

$$\left(\sum_{j=1}^{n} f_j C_j\right)[1 - pK] = K. \quad \text{(A23)}$$

Hence, we have $\sum_{j=1}^{n} f_j C_j = K/(1 - pK)$. By substituting this quantity back into the right-hand side of equation (A21) we obtain C_i as in equation (A19).

For claim (b), suppose that $K < 1/p$ and $\hat{\theta}_i + p > 0$ for all $i = 1,\dots,n$; then equation (A19) implies that $C_i > 0$ for all i. Conversely, suppose that $C_i > 0$ for all i. From equation (A14) we see that this implies $\hat{\theta}_i + p > 0$ for all i. Given this result, from equation (A19) we have that $C_i > 0$ for all i also implies $K < 1/p$.

To prove claim (c), we see that condition (A20) clearly implies $\hat{\theta}_i + p > 0$ for all $i = 1,\dots,n$. Moreover, from the definition of K we have

$$K = \sum_{i=1}^{n} \frac{f_i}{\hat{\theta}_i + p} < \sum_{i=1}^{n} \frac{f_i}{\min_{i=1,\dots,n} \{\hat{\theta}_i\} + p} = \frac{1}{\min_{i=1,\dots,n} \{\hat{\theta}_i\} + p} < \frac{1}{p};$$

the last inequality stems from $\min_{i=1,\dots,n}\{\hat{\theta}_i\} > 0$ because of condition (A20). Q.E.D.

Proof of Lemma 3: (a) For given vector $(\pi_1,\dots,\pi_n)$, $\sum_{j=1}^{n} \pi_j C_j$ is a constant. Moreover, because for $\gamma > 1$, C_i is monotonically strictly decreasing as i increases from 1 to n, we must have $C_i/\sum_{j=1}^{n} \pi_j C_j$ monotonically decreasing with i. Since we must also have $C_1/\sum_{j=1}^{n} \pi_j C_j > 1 > C_n/\sum_{j=1}^{n} \pi_j C_j$, it follows that there is a k such that $\pi_i^* = (C_i/\sum_{j=1}^{n} \pi_j C_j)\pi_i > \pi_i$ for $i < k$ and $\pi_i^* = (C_i/\sum_{j=1}^{n} \pi_j C_j)\pi_i < \pi_i$ for $i \geq k$. Since $\theta_1 < \theta_2 < \cdots < \theta_n$, the average of the θ_is using the distribution $(\pi_1^*,\dots,\pi_n^*)$ must be smaller than using the distribution $(\pi_1,\dots,\pi_n)$ because the former gives more weight to low θ_is and less to high θ_is than the latter distribution; that is, $m_\theta^* < m_\theta$. Hence, $V_\theta = m_\theta^* - m_\theta < 0$ if $\gamma > 1$. The reverse argument holds for $\gamma < 1$, in which case C_i is monotonically strictly increasing in i.

(b) Consider a mean-preserving spread s_i on the distribution Π (see footnote 2) and denote the new distribution by $\tilde{\Pi}$. We want to show that $\tilde{m}_\theta^* < m_\theta^*$ if $\gamma > 1$ and $\tilde{m}_\theta^* > m_\theta^*$ if $\gamma < 1$, where $\tilde{m}_\theta^*$ is the mean growth rate computed using the probability $\tilde{\Pi}$.

Step 1: For $\gamma > 1$, define the function $g_1(x) = C(x) \times (x - \theta_1)$ on the real interval $[\theta_1,\theta_n]$, where $C(x)$ is given by

$$C(x) = \frac{1}{[(\gamma - 1)x + A]B}, \tag{A24}$$

where, from equation (17), $A = \phi + p + \frac{1}{2}\gamma(1 - \gamma)\sigma_D^2$ and $B = (1 - pK)$. I first show that $g_1(x)$ is a concave function of $x \in [\theta_1,\theta_n]$. By differentiating g_1 twice with respect to x, we obtain

$$g_1''(x) = -\frac{2(\gamma - 1)[(\gamma - 1)\theta_1 + A]}{[(\gamma - 1)x + A]^3 B} < 0 \Leftrightarrow \gamma > 1 \tag{A25}$$

because from condition (A14) we have $[(\gamma - 1)x + A] > 0$ for all $x \in [\theta_1,\theta_n]$.

Step 2: Since the expected value of a convex (concave) function increases (decreases) after a mean-preserving spread has been performed on the underlying probability distribution (see, e.g., Ingersoll (1987, p. 116)), we have $\sum_{i=1}^{n} \pi_i g_1(\theta_i) > \sum_{i=1}^{n} \tilde{\pi}_i g_1(\theta_i) > 0$ and $\sum_{j=1}^{n} \pi_j C_j < \sum_{j=1}^{n} \tilde{\pi}_j C_j$. Hence,

$$\begin{aligned}
\tilde{m}_\theta^* &= \sum_{i=1}^{n} \tilde{\pi}_i^* \theta_i = \sum_{i=1}^{n} \tilde{\pi}_i^* \theta_i - \theta_1 + \theta_1 = \sum_{i=1}^{n} \tilde{\pi}_i^*(\theta_i - \theta_1) + \theta_1 \\
&= \sum_{i=1}^{n} \frac{\tilde{\pi}_i C_i(\theta_i - \theta_1)}{\sum_{j=1}^{n} \tilde{\pi}_j C_j} + \theta_1 = \frac{\sum_{i=1}^{n} \tilde{\pi}_i g_1(\theta_i)}{\sum_{j=1}^{n} \tilde{\pi}_j C_j} + \theta_1 \\
&< \frac{\sum_{i=1}^{n} \pi_i g_1(\theta_i)}{\sum_{j=1}^{n} \pi_j C_j} + \theta_1 = \sum_{i=1}^{n} \frac{\pi_i C_i(\theta_i - \theta_1)}{\sum_{j=1}^{n} \pi_j C_j} + \theta_1 \\
&= \sum_{i=1}^{n} \pi_i^*(\theta_i - \theta_1) + \theta_1 = \sum_{i=1}^{n} \pi_i^* \theta_i = m_\theta^*.
\end{aligned} \tag{A26}$$

Step 3: For $\gamma < 1$, let $g_2(x) = C(x) \times (x - \theta_n)$. Hence,

$$g_2''(x) = -\frac{2(\gamma - 1)[(\gamma - 1)\theta_n + A]}{[(\gamma - 1)x + A]^3 B} > 0 \Leftrightarrow \gamma < 1.$$

Finally, as before we can write

$$\begin{aligned}
\tilde{m}_\theta^* &= \sum_{i=1}^{n} \tilde{\pi}_i^* \theta_i = \sum_{i=1}^{n} \frac{\tilde{\pi}_i C_i(\theta_i - \theta_n)}{\sum_{j=1}^{n} \tilde{\pi}_j C_j} + \theta_n = \frac{\sum_{i=1}^{n} \tilde{\pi}_i g_2(\theta_i)}{\sum_{j=1}^{n} \tilde{\pi}_j C_j} + \theta_n \\
&> \frac{\sum_{i=1}^{n} \pi_i g_2(\theta_i)}{\sum_{j=1}^{n} \pi_j C_j} + \theta_n = \sum_{i=1}^{n} \frac{\pi_i C_i(\theta_i - \theta_n)}{\sum_{j=1}^{n} \pi_j C_j} + \theta_n \\
&= \sum_{i=1}^{n} \pi_i^* \theta_i = m_\theta^*.
\end{aligned} \tag{A27}$$

Now the inequality stems from $\sum_{i=1}^{n} \pi_i g_2(\theta_i) < \sum_{i=1}^{n} \tilde{\pi}_i g_2(\theta_i) < 0$ and $\sum_{j=1}^{n} \pi_j C_j < \sum_{j=1}^{n} \tilde{\pi}_j C_j$.

(c) For $\gamma \neq 1$, let

$$\varepsilon(\gamma) = \frac{\phi + p}{\gamma - 1} - \frac{1}{2}\gamma\sigma_D^2 + \theta_1.$$

The condition (A14) requires $\varepsilon(\gamma) \neq 0$. Of course

$$\frac{d\varepsilon}{d\gamma} = -\frac{\phi + p}{(\gamma - 1)^2} - \frac{1}{2}\sigma_D^2 < 0.$$

From equation (A19) we can rewrite

$$C(\theta) = C_1 \frac{\varepsilon(\gamma)}{\theta - \theta_1 + \varepsilon(\gamma)},$$

and therefore

$$m_\theta^* = \frac{\sum_{i=1}^{n} \pi_i C_i \theta_i}{\sum_{i=1}^{n} \pi_i C_i} = \frac{\sum_{i=1}^{n} \frac{\pi_i \theta_i}{\theta_i - \theta_1 + \varepsilon(\gamma)}}{\sum_{i=1}^{n} \frac{\pi_i}{\theta_i - \theta_1 + \varepsilon(\gamma)}}. \tag{A28}$$

Hence,

$$\frac{dm_\theta^*}{d\varepsilon} = \frac{\left(\sum_{i=1}^{n} \frac{-\pi_i\theta_i}{(\theta_i - \theta_1 + \varepsilon)^2}\right)\left(\sum_{i=1}^{n} \frac{\pi_i}{\theta_i - \theta_1 + \varepsilon}\right) + \left(\sum_{i=1}^{n} \frac{\pi_i\theta_i}{\theta_i - \theta_1 + \varepsilon}\right)\left(\sum_{i=1}^{n} \frac{\pi_i}{(\theta_i - \theta_1 + \varepsilon)^2}\right)}{\left(\sum_{i=1}^{n} \frac{\pi_i}{\theta_i - \theta_1 + \varepsilon}\right)^2}, \tag{A29}$$

which is positive if and only if

$$\frac{\sum_{i=1}^{n} \frac{\pi_i\theta_i}{(\theta_i - \theta_1 + \varepsilon)^2}}{\sum_{i=1}^{n} \frac{\pi_i}{(\theta_i - \theta_1 + \varepsilon)^2}} < \frac{\sum_{i=1}^{n} \frac{\pi_i\theta_i}{\theta_i - \theta_1 + \varepsilon}}{\sum_{i=1}^{n} \frac{\pi_i}{\theta_i - \theta_1 + \varepsilon}}.$$

Notice that the right-hand side is simply m_θ^* and the left-hand side can be written as $m_\theta^{**} = \sum_{i=1}^n \pi_i^{**}\theta_i$, where π_i^{**} is defined by

$$\pi_i^{**} = \frac{\dfrac{\pi_i}{(\theta_i - \theta_1 + \varepsilon)^2}}{\displaystyle\sum_{i=1}^n \frac{\pi_i}{(\theta_i - \theta_1 + \varepsilon)^2}}$$

$$= \frac{1}{(\theta_i - \theta_1 + \varepsilon)}\left(\frac{\dfrac{\pi_i}{\theta_i - \theta_1 + \varepsilon}}{\displaystyle\sum_{i=1}^n \frac{\pi_i}{\theta_i - \theta_1 + \varepsilon}}\right)\left(\frac{\displaystyle\sum_{i=1}^n \frac{\pi_i}{\theta_i - \theta_1 + \varepsilon}}{\displaystyle\sum_{i=1}^n \frac{\pi_i}{(\theta_i - \theta_1 + \varepsilon)^2}}\right) \tag{A30}$$

$$= \frac{\pi_i^*}{(\theta_i - \theta_1 + \varepsilon)}\left(\frac{\displaystyle\sum_{i=1}^n \frac{\pi_i}{\theta_i - \theta_1 + \varepsilon}}{\displaystyle\sum_{i=1}^n \frac{\pi_i}{(\theta_i - \theta_1 + \varepsilon)^2}}\right).$$

Since the last parenthetical expression is independent of i and since $(\theta_i - \theta_1 + \varepsilon)$ is increasing in i, it must be the case that $\pi_i^{**} > \pi_i^*$ for low θ_is and $\pi_i^{**} < \pi_i^*$ for high θ_is. Therefore, $m_\theta^{**} < m_\theta^*$, and hence $dm_\theta^*/d\varepsilon > 0$. Now

$$\frac{dV_\theta}{d\gamma} = \frac{dm_\theta^*}{d\gamma} = \frac{dm_\theta^*}{d\varepsilon}\frac{d\varepsilon(\gamma)}{d\gamma} < 0. \tag{A31}$$

Finally, since V_θ is continuous at $\gamma = 1$ and $V_\theta > 0$ for $\gamma < 1$ and $V_\theta < 0$ for $\gamma > 1$, the function V_θ must be weakly decreasing for all γ. Q.E.D.

Proof of Proposition 3: (a) is immediate from Lemma 3(b) and $\mu_R = \gamma(\sigma_D^2 + V_\theta)$.

(b) From the proof of Proposition 2(c), we know that $\varepsilon(\gamma) > 0$ and $\varepsilon(\gamma) \to 0$ as $\gamma \to \gamma^*$, the upper bound (greater than one) implied by the condition (A14). Notice that from $\mu_R = \gamma(\sigma_D^2 + V_\theta)$ and the proof of Lemma 3, we obtain

$$\frac{d\mu_R}{d\gamma} = \sigma_D^2 + V_\theta + \gamma\frac{dV_\theta}{d\gamma} = \sigma_D^2 + V_\theta - \gamma\left(\frac{\phi + p}{(\gamma - 1)^2} + \frac{1}{2}\sigma_D^2\right)\left(\frac{dm_\theta^*}{d\varepsilon}\right). \tag{A32}$$

I first show that as $\varepsilon(\gamma) \to 0$ we have (i) $V_\theta = m_\theta^* - m_\theta \to \theta_1 - m_\theta$, and (ii) $(dm_\theta^*/d\varepsilon) \to ((1 - \pi_1)/\pi_1)$.

For (i) we can write

$$\pi_i^* = \frac{\pi_i C_i}{\sum_{j=1}^{n} \pi_j C_j} = \frac{\dfrac{\pi_i}{\theta_i - \theta_1 + \varepsilon(\gamma)}}{\sum_{j=1}^{n} \dfrac{\pi_j}{\theta_j - \theta_1 + \varepsilon(\gamma)}}. \tag{A33}$$

Notice that since $\pi_i > 0$ for all is, as $\varepsilon(\gamma) \to 0$, $\pi_1^* \to 1$ and $\pi_i^* \to 0$ for $i \neq 1$, and hence $V_\theta = m_\theta^* - m_\theta \to \theta_1 - m_\theta$.

For (ii) consider the first parenthetical expression in the numerator of equation (A29). We can write it as

$$\left(\sum_{i=1}^{n} \frac{-\pi_i\theta_i}{(\theta_i - \theta_1 + \varepsilon)^2}\right) = \left(-\frac{\pi_1\theta_1}{\varepsilon^2} + \sum_{i=2}^{n} \frac{-\pi_i\theta_i}{(\theta_i - \theta_1 + \varepsilon)^2}\right)$$
$$= \frac{1}{\varepsilon^2}\left(-\pi_1\theta_1 + \varepsilon^2 \sum_{i=2}^{n} \frac{-\pi_i\theta_i}{(\theta_i - \theta_1 + \varepsilon)^2}\right).$$

By doing the same exercise for all the other terms, we obtain

$$\frac{dm_\theta^*}{d\varepsilon} = \frac{\dfrac{1}{\varepsilon}}{\left(\pi_1 + \varepsilon \sum_{i=2}^{n} \dfrac{\pi_i}{\theta_i - \theta_1 + \varepsilon}\right)^2}$$
$$\times \left\{\left(-\pi_1\theta_1 + \varepsilon^2 \sum_{i=2}^{n} \frac{-\pi_i\theta_i}{(\theta_i - \theta_1 + \varepsilon)^2}\right)\left(\pi_1 + \varepsilon \sum_{i=2}^{n} \frac{\pi_i}{\theta_i - \theta_1 + \varepsilon}\right)\right.$$
$$\left. + \left(\pi_1\theta_1 + \varepsilon \sum_{i=2}^{n} \frac{\pi_i\theta_i}{\theta_i - \theta_1 + \varepsilon}\right)\left(\pi_1 + \varepsilon^2 \sum_{i=2}^{n} \frac{\pi_i}{(\theta_i - \theta_1 + \varepsilon)^2}\right)\right\}$$
$$= \frac{1}{\left(\pi_1 + \varepsilon \sum_{i=2}^{n} \dfrac{\pi_i}{\theta_i - \theta_1 + \varepsilon}\right)^2} \tag{A34}$$
$$\times \left\{-\pi_1\theta_1 \sum_{i=2}^{n} \frac{\pi_i}{\theta_i - \theta_1 + \varepsilon} + \varepsilon \sum_{i=2}^{n} \frac{-\pi_i\theta_i}{(\theta_i - \theta_1 + \varepsilon)^2}\left(\pi_1 + \varepsilon \sum_{i=2}^{n} \frac{\pi_i}{\theta_i - \theta_1 + \varepsilon}\right)\right.$$
$$\left. + \pi_1 \sum_{i=2}^{n} \frac{\pi_i\theta_i}{\theta_i - \theta_1 + \varepsilon} + \varepsilon \sum_{i=2}^{n} \frac{\pi_i}{(\theta_i - \theta_1 + \varepsilon)^2}\left(\pi_1\theta_1 + \varepsilon \sum_{i=2}^{n} \frac{\pi_i\theta_i}{\theta_i - \theta_1 + \varepsilon}\right)\right\}$$
$$\to \frac{\sum_{i=2}^{n} \dfrac{\pi_i(\theta_i - \theta_1)}{\theta_i - \theta_1}}{\pi_1} = \frac{1 - \pi_1}{\pi_1}.$$

To conclude the proof of Proposition 3(b), notice that if $m_\theta > \sigma_D^2 + \theta_1$ then, from equation (A32), $(dm_\theta^*/d\varepsilon) > 0$, and the result in part (i), a continuity argument yields the result.

Similarly, if

$$\pi_1 < \bar{\pi}_1 = \frac{\gamma^*\left(\dfrac{\phi + p}{(\gamma^* - 1)^2} + \dfrac{1}{2}\sigma_D^2\right)}{\sigma_D^2 + \gamma^*\left(\dfrac{\phi + p}{(\gamma^* - 1)^2} + \dfrac{1}{2}\sigma_D^2\right)}, \tag{A35}$$

where again γ^* is the upper bound of γ implied by the condition (A14), then

$$\sigma_D^2 - \gamma^*\left(\frac{\phi + p}{(\gamma^* - 1)^2} + \frac{1}{2}\sigma_D^2\right)\frac{1 - \pi_1}{\pi_1} < 0.$$

Hence from equation (A32) and the result of part (ii), a continuity argument yields the result.

(c) The result follows from $\mu_R = \gamma(\sigma_D^2 + V_\theta)$ and the fact that $V_\theta = m_\theta^* - m_\theta \to \theta_1 - m_\theta$. Indeed, if $m_\theta > \sigma_D^2 + \theta_1$ by continuity there is a $\bar{\gamma}$ such that $\mu_R < 0$ for $\gamma > \bar{\gamma}$. Moreover, for a given γ, part (a) shows that a mean-preserving spread decreases V_θ. Hence, a negative risk premium can be obtained for a lower risk aversion coefficient when the distribution is more diffused. Q.E.D.

Proof of Proposition 4: Part (a) is immediate from the fact that σ_R^2 is a parabola with respect to V_θ and the fact that V_θ is monotonic decreasing in γ.

For part (b) the claims holds if $2 + (h_e^2 + h_D^2)V_\theta < 0$ for a high enough γ. From the proof of Proposition 3(c), as γ increases, $V_\theta \to \theta_1 - m_\theta$. Hence, by continuity, it is sufficient to show that $2 + (h_e^2 + h_D^2)(\theta_1 - m_\theta) < 0$; that is,

$$m_\theta - \theta_1 > \frac{2}{h_e^2 + h_D^2} = \sigma_D^2\left(\frac{2\sigma_e^2}{\sigma_D^2 + \sigma_e^2}\right). \tag{A36}$$

This condition is satisfied if $m_\theta > \sigma_D^2 + \theta_1$ (the condition in Proposition 3(c)) whenever the expression in parentheses is less than one—that is, when $h_e^2 > h_D^2$. Q.E.D.

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